THE EFFECT OF TWO DIMENSIONAL NONUNIFORMITY OF THE INITIAL PARAMETERS ON PLASMA INSTABILITY

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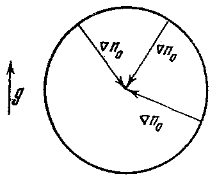
1. It is well known that new and unsafe instabilities associated with the so called drift oscillations (see, e.g., [1, 2]) appear in a nonuniform plasma. The properties of these instabilities have been fairly thoroughly studied for the case in which the initial plasma irregularity is one dimensional. However, in real conditions it is not always possible to reduce the problem to a one-dimensional case. By way of an example we shall consider a situation typical for the so called drift-thermal instabilities (see, e.g., [3, 4]). These instabilities often develop when there is a specific relation between the initial gradients of density n_0 and temperature T_0. At the present time the most unstable of these is taken to be [3] the instability which develops for $d \lg T_0 / d \lg n_0 > 2$ in the frequency region $\omega < k_z v_{Ti}$; here v_{Ti} is the ion thermal velocity, k_z is the wave vector component in the direction of the magnetic field H_0.



Its instability is connected with the fact that the anomalous particle diffusion brought about by a developing unstable configuration remains large even in devices which produce a stabilizing factor such as crossed lines of force in the magnetic field [5].

In paper [5] mention was made of the difficulty of creating experimental conditions so as to avoid the unstable range of temperature and density gradients since the total particle flux to the wall of the device is less than the thermal flux. Naturally these conditions can only deteriorate if for some reason the unstable region of gradients should increase.

It turns out that a possible noncollinearity of the temperature and density gradients may lead to a broadening of the instability region. This possibility may be confirmed on a series of counts. Firstly, no real devices have such a high degree of symmetry that the density and temperature gradients are only radial. Then it does not by any means follow from the general conditions for equilibrium that ∇n_0 and ∇T_0 will necessarily be collinear.

Secondly, additional particle and heat fluxes may arise during the operation of the device and these may dissipate across the field only with difficulty since the plasma is magnetized.

Finally, peculiarities in the construction even of symmetrical devices may lead to the appearance not only of radial but also of azimuthal gradients. The stellarator is an example of this.

The figure gives a cross section of a stellarator perpendicular to the lines of force of H_0. Since the lines of force are the surface of a torus in the first approximation, then because of their curvature the effective force of gravity is in a certain direction and makes a varying angle with the radial density gradient.* On the other hand gravity also has a component in the azimuthal direction, and it then follows from the conditions for equilibrium that there should be an azimuthal density gradient. We note that the initial nonuniformity now takes on a marked two-dimensional character.

2. In deriving the dispersion equation we will allow for the dependence of density and temperature on the coordinates x, y, i.e., $n_0(x, y)$, $T_0(x, y)$. The two dimensional character of the initial nonuniformity leads to a change in the drift frequencies. Actually if we set $n_0 = n_0(x, y)$, $T_0 = \text{const}$, then the equations* describing the drift wave take the following form for perturbations of the order $\exp(-i\omega t + ikr)$:

$$-i\omega n + c\frac{E_y}{H_0}\frac{\partial n_0}{\partial x} - c\frac{E_x}{H_0}\frac{\partial n_0}{\partial y} = 0,$$

$$-ik_z n T_0 - e n_0 E_z = 0. \tag{2.1}$$

Equations (2.1) describe the electron equilibrium along the lines of magnetic field H_0 and the conservation of particles. Here n is the perturbed density, c is the velocity of light, $E_{x,y,z}$ are the electric field components. From (2.1) we have

$$\omega = \frac{cT_0}{eH_0 n_0}\left(k_y \frac{\partial n_0}{\partial x} - k_x \frac{\partial n_0}{\partial y}\right) \equiv \omega_i. \tag{2.2}$$

For the initial equation describing charge motion in the general case we shall take the kinetic equation for a perturbed correction to the distribution function without a collision integral:

$$\frac{\partial f_j}{\partial t} + (\mathbf{v}\nabla) f_j + \omega_{Hj}\left[\mathbf{v} \times \frac{\mathbf{H}_0}{H_0}\right]\frac{\partial f_j}{\partial \mathbf{v}} + \frac{e_j}{m_j}\left(\mathbf{E}\,\frac{\partial f_j^0}{\partial \mathbf{v}}\right) = 0. \tag{2.3}$$

Here ω_{Hj} is the Larmor frequency of j-type particles.

The equilibrium distribution function f_j^0 should satisfy the equation

$$(\mathbf{v}\nabla\cdot) f_j^0 + \omega_{Hj}\left[\mathbf{v} \times \frac{\mathbf{H}_0}{H_0}\right]\frac{\partial f_j^0}{\partial \mathbf{v}} = 0. \tag{2.4}$$

The solution of the equation will be an arbitrary function of the integrals of motion given by the equations

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{v}}{dt} = \omega_{Hj}\left[\mathbf{v} \times \frac{\mathbf{H}_0}{H_0}\right]. \tag{2.5}$$

The first integrals of this system are

$$\varepsilon, \qquad \left(x + \frac{v_y}{\omega_{Hj}}\right), \qquad \left(y - \frac{v_x}{\omega_{Hj}}\right).$$

Here ε is the energy. It is natural to choose the distribution function of the unperturbed plasma in the form

$$f_j^0 = \left[1 + \left(x + \frac{v_y}{\omega_{Hj}}\right)\frac{\partial}{\partial x} + \left(y - \frac{v_x}{\omega_{Hj}}\right)\frac{\partial}{\partial y}\right] f_{0j},$$

$$f_{0j} = n_0\left(\frac{m_j}{2\pi T_{0j}}\right)^{3/2} \exp\left(-\frac{m_j v^2}{2T_{0j}}\right). \tag{2.6}$$

We note that previously only distribution functions were treated which depended on ε and $(x + v_y/\omega_{Hj})$. The solution of Eq. (2.3) has the form

$$f_j = -\frac{e_j}{m_j}\int_{-\infty}^{t}\left(\mathbf{E}\,\frac{\partial f_j^0}{\partial \mathbf{v}}\right) dt\,. \tag{2.7}$$

Here the integral is taken over the unperturbed trajectories of j-type particles specified by Eqs. (2.5).

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*R. Z. Sagdeev pointed out that this transverse corrugation exists naturally in the stellarator.

*Here and in what follows it is assumed that the initial parameters change slowly enough for the quasi-classical approximation to be applicable [6].

The dispersion equation $n_e = n_i$ (n_j is the perturbed density of j-type particles) takes the form

$$\int_{-\infty}^{\infty} f_e d^3v = \int_{-\infty}^{\infty} f_i d^3v. \tag{2.8}$$

Omitting the intermediate calculations, similar to those of [3] for example, we give only the final form of Eq. (2.8):

$$\begin{gathered}\frac{2n_0}{T_0} - \sum_j \sum_{l=0}^{\infty} \left(\frac{\omega}{T_0} - \frac{k_y}{m_j \omega_{Hj}} \frac{\partial}{\partial x} + \frac{k_x}{m_j \omega_{Hj}} \frac{\partial}{\partial y} \right) \times \\ \times \int_{-\infty}^{\infty} \frac{f_{0j}(v_z)\, dv_z}{\omega - k_z v_z - l\omega_{Hj}} \int_0^{\infty} J^2 \left(\frac{k_\perp v_\perp}{\omega_{Hj}} \right) \times \\ \times \exp\left(-\frac{m_j v_\perp^2}{2T_{0j}} \right) \frac{m_j v_\perp}{T_{0j}} dv_\perp = 0. \end{gathered} \tag{2.9}$$

When $\omega \ll \omega_{Hj}$ it suffices to retain only the term with $l = 0$ in the sum over l, allowing for the fact that

$$\begin{gathered}\int_0^{\infty} \exp\left(-\frac{m_j v_\perp^2}{2T_{0j}} \right) J^2 \left(\frac{k_\perp v_\perp}{\omega_{Hj}} \right) \frac{m_j v_\perp}{T_{0j}} dv_\perp = \\ = I_0 \left(\frac{\theta_j^2}{2} \right) \exp\left(-\frac{\theta_j^2}{2} \right) \quad \left(\theta_j = \frac{k_\perp v_{Tj}}{\omega_{Hj}} \right);\end{gathered}$$

here I_0 is a Bessel function of imaginary argument, v_{Tj} is the thermal velocity of the particles. Then Eq. (2.9) may be rewritten in the following form:

$$\begin{gathered} 2 - A \frac{\omega_T \omega}{(k_z v_{Ti})^2} + \frac{i\sqrt{\pi}}{k_z v_{Ti}} \left(1 + \frac{2}{\sqrt{\pi}} i \frac{\omega}{k_z v_{Ti}} \right) \times \\ \times \left[\omega - \omega_i + \frac{1}{2} \omega_T (1 + \delta) \right] A = 0, \\ \omega_T = \frac{c}{eH_0} \left(k_y \frac{\partial T_0}{\partial x} - k_x \frac{\partial T_0}{\partial y} \right), \\ A = \exp\left(-\frac{\theta_i^2}{2} \right) I_0 \left(\frac{\theta_i^2}{2} \right), \quad \delta = \theta_i^2 \left[1 - \frac{I_1(^1/_2\, \theta_i^2)}{I_0(^1/_2\, \theta_i^2)} \right]. \end{gathered} \tag{2.10}$$

In Eq. (2.10) the contribution from electron currents is neglected, which is legitimate for $k_\perp v_{Tj} / \omega_{Hj} \ll 1$, and allowance is made for the fact that $\omega < k_z v_{Ti}$. Solving Eq. (2.10) we find

$$\begin{gathered} \operatorname{Im} \omega = -ik_z v_{Ti} \left\langle \frac{2}{A} \sqrt{\pi} + \sqrt{\pi} \frac{\omega_i^2}{(k_z v_{Ti})^2} \times \right. \\ \times \left[\frac{1}{2} \frac{\alpha}{\beta} (1 + \delta) - 1 \right] \times \\ \left. \times \left\{ \frac{\alpha}{\beta} + 2 \left[\frac{1}{2} \frac{\alpha}{\beta} (1 + \delta) - 1 \right] \right\} \right\rangle \left\{ \frac{\omega_i^2}{(k_z v_{Ti})^2} \times \right. \\ \left. \times \left[\frac{\alpha}{\beta} + 2 \left(\frac{1}{2} \frac{\alpha}{\beta} (1 + \delta) - 1 \right) \right]^2 + \pi \right\}^{-1}, \\ \alpha = \frac{1}{T_0} \left(k_y \frac{\partial T_0}{\partial x} - k_x \frac{\partial T_0}{\partial y} \right), \\ \beta = \frac{1}{n_0} \left(k_y \frac{\partial n_0}{\partial x} - k_x \frac{\partial n_0}{\partial y} \right). \end{gathered} \tag{2.11}$$

In accordance with (2.10) the condition for instability $\operatorname{Im}\omega > 0$ assumes the form

$$\frac{n_0 \partial T_0 / \partial x}{T_0 \partial n_0 / \partial x} > \frac{2}{1 + \delta} \left(1 - \frac{k_x}{k_y} \frac{\partial n_0 / \partial y}{\partial n_0 / \partial x} \right) \left(1 - \frac{k_x}{k_y} \frac{\partial T_0 / \partial y}{\partial T_0 / \partial x} \right)^{-1}. \tag{2.12}$$

It follows from this inequality that if ∇n_0 and ∇T_0 are not collinear, then the instability region is broadened considerably compared with the region indicated in paper [3]. The instability region now includes the case when $\partial T_0/\partial x$ and $\partial n_0/\partial x$ have different signs; if $\partial T_0/\partial y = 0$, then instability is possible for

$$\vartheta = \frac{\partial \lg T_0 / \partial x}{\partial \lg n_0 / \partial x} \ll 1.$$

Here $k_x \gg k_y$ if $\partial n_0/\partial y \ll \partial n_0/\partial x$. It is necessary to bear in mind, however, that the minimum value of β which may yet be attained must not violate the condition $\omega_i \gtrsim k_z v_{Ti}$, since

$$\operatorname{Re} \omega \sim k_z^2 v_{Ti}^2 / \omega_i < k_z v_{Ti}.$$

It should be noted that noncollinearity of ∇T_0 and ∇n_0 leads to a change of the instability region in a series of other cases. It is not hard to see, for example, that the instability region broadens [3] when the thermal conductivity along the lines of force of H_0 is allowed for; the boundary of the region is shifted substantially in the direction of positive ϑ.

In fact it follows from the considerations adduced above that the criterion for the development of drift-temperature instabilities may be written in the form of a relation between ω_i, as determined from formula (2.2) and the quantity ω_T determined in a similar manner:

$$\omega_T = \frac{c}{eH_0} \left(k_y \frac{\partial T_0}{\partial x} - k_x \frac{\partial T_0}{\partial y} \right). \tag{2.13}$$

The criterion for development of the instability associated with allowing for the thermal conductivity along H_0 may then be written in the form

$$\omega_T / \omega_i < 0. \tag{2.14}$$

In the case in which ∇T_0 and ∇n_0 are collinear we have the criterion [3] $d \lg T_0 / d \lg n_0 < 0$. In the contrary case we have

$$\frac{(k_y\, \partial T_0 / \partial x - k_x\, \partial T_0 / \partial y)}{k_y\, \partial n_0 / \partial x} < 0. \tag{2.15}$$

For simplicity expression (2.15) has been written for the case $\partial n_0/\partial y = 0$. The assertion made above follows from this.

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